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EXACT SOLUTION OF SOME LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

An exact real solution of linear Volterra – Fredholm and Volterra loaded integro-differential equation $Bx = f$ is presented.

Keywords: integro-differential equations, linear Volterra – Fredholm integro-differential equation, linear Volterra loaded integro-differential equation, exact solution, boundary value problem.

Introduction. Volterra – Fredholm integro-differential equations (IDEs) appear in various fields of science such as physics, biology, and engineering. Such equations are the mathematical models in the problems of mechanics, electromagnetic theory, population dynamics, fluid dynamics, pharmacokinetic studies and many others [1–5]. Usually it is difficult to solve Volterra – Fredholm IDE using analytical methods. By Adomian decomposition method in [6–7] the solution of such IDEs is given in an infinite series of components that can be recurrently determined. Spline collocation method was used to solve Volterra integro-differential equations in [8]. The numerical solution of linear Fredholm – Volterra integro-differential equations was studied in [9]. In [10] was developed an existence and uniqueness theorem of the solution to an initial value problems for a class of second-order impulsive integro-differential equations of Volterra type in a real Banach space by using the generalized Banach fixed point theorem. Existence and Uniqueness Theorems are also illustrated for the systems of Volterra IDEs of first order in [2], [11–14] and for Volterra IDEs of the first and second order in [6–7, 20, 22]. Exact solutions to Volterra – Fredholm integro-differential equations were obtained in [4, 19]. We investigate the problem

$$Bu(x) = \widehat{A}u(x) - g(x)\Phi(u) = f(x),\ D(B) = D\left(\widehat{A}\right), \tag{1}$$

where an operator B contains a linear differential operator $\widehat{A}$, the linear Volterra integral operator K with kernel of convolution type and the inner product of vectors $g(x)\Phi(u)$ with linear bounded functionals Φ_i, $i = 1, ..., m$. Similar problem for arbitrary Banach spaces X, Y and Fredholm integro-differential (loaded differential) equation $Bu(x) = \widehat{A}u(x) - g(x)\Phi(u) = f(x)$, $D(B) = D(\widehat{A})$, where $\widehat{A}$ is a correct operator and $\Phi_i(u)$ are bounded integral functionals (loaded part of differential equations), was investigated in [20] and for $Bu(x) = f(x)$, with nonlocal boundary conditions in [21]. We prove in this paper that the operator $\widehat{A} - K$ is continuously invertible and, using Oinarov extensions of linear operators in Banach space [22], we obtain exact real solutions in closed form of Problem (1). The technique which we present, is simple to use and can be easily incorporated to any Computer Algebra System (CAS).

Terminology and notation. Let X be a complex Banach space and X^* its adjoint space, i.e. the set of all complex-valued linear and bounded functionals on X. We denote by $f(x)$ the value of f on x. We write $D(A)$ and $R(A)$ for the domain and the range of the operator A, respectively. A linear operator B is said to be an *injective operator*, if $\ker B = \{0\}$. An operator A is called *invertible* if there exists the inverse operator A^{-1}. An operator A is called *continuously invertible* if it is invertible and the inverse operator A^{-1} is continuous. Remind that if a linear operator is injective, then it is invertible. A bounded operator $B: X \to Y$ is said to be *compact* if the image $\{Bu_n\}$ of any bounded sequence $\{u_n\}$ of X contains a Cauchy subsequence. A function $f(x)$ is said to be of *exponential order* α if $|f(x)| \le Me^{\alpha x}$, $0 \le x < \infty$, where M and α are constants. By $F(s) = \mathrm{L}[f(x)]$ we denote Laplace transform of function $f(x)$ and by $\mathrm{L}^{-1}[F(s)] = f(x)$ the inverse Laplace transform of $F(s)$. An equation is called *loaded equation* if it contains the solution function on a manifold with dimension less than the dimension of domain of this function [23]. For example an ordinary loaded differential equation is represented by

$$dy/dx = f(x, y) + \psi\big(y(x_j)\big),\ x \in [0,1],\ x_j \in [0,1],$$

where x_j are fixed points. If $\Phi_i \in X^*$, $i = 1, ..., m$, then we denote by $\Phi = \mathrm{col}(\Phi_1, ..., \Phi_m)$ and $\Phi(x) = \mathrm{col}(\Phi_1(x), ..., \Phi_m(x))$. Let $g = (g_1, ..., g_m)$ be a vector of X^m. We will denote by $\Phi(g)$ the $m \times m$ matrix whose *i,j*-th entry is the value of functional Φ_i on element g_j and by I_m the identity $m \times m$ matrix, respectively. Note that $\Phi(gC) = \Phi(g)C$, where C is a $m\times m$ constant matrix. By **c** and **0** we will denote a vector and zero vector, respectively. We also denote $u = u(x)$, $g = g(x)$, $f = f(x)$, $\Omega = [a, b] \times [a, b]$, $C_0[a, b] = \{u(x)\in C[a; b]: u(a) = 0\}$, $C_0^n[a; b] = \{u(x) \in C^n[a; b]: u^{(j-1)}(a) = 0, j = 1, \ldots, n\}$, $\|u\|_{C_0^n} = \|u\|_{C^n} = \sum_{k=0}^{n} \left\|u^{(k)}\right\|_C$ and

$$\widehat{A}u(x) = \sum_{i=0}^{n} \alpha_i u^{(n-i)}(x),\ \alpha_n \neq 0,\ D(\widehat{A}) = C_0^n[a,b],$$

$$Ku(x) = \sum_{j=0}^{n} \int_a^x K_j(x-t) u^{(j)}(t)\,dt, K_j(x-t) \in C(\Omega),\ D(K) = C^n[a,b],$$

$$g\Phi(u)(x) = g(x)\Phi(u) = \sum_{k=1}^{m} g_k(x)\Phi_k(u),$$

where $\alpha_0, \alpha_1, \ldots, \alpha_n$ are constants, $g_1(x), \ldots, g_m(x)$ are linearly independent in $C[a,b]$ functions, $g(x) = (g_1(x), \ldots, g_m(x))$, $\Phi_1, \ldots, \Phi_m$ the set of bounded linearly independent on $C^{n-1}[a,b]$ functionals, $\Phi(u) = \mathrm{col}(\Phi_1(u), \ldots, \Phi_m(u))$ and $K_j(0) = 0$, $j = 0,1, \ldots, n$. Denote $K_{j,0}(z) = K_j(z)$, $j = 0,1, \ldots, n$. Let $K_{j,i+1}(z)$ be the antiderivative of $K_{j,i}(z)$, such that $K_{j,i}(0) = 0$, $j = 0,1, \ldots, n$, $i = 0,1, \ldots, n-1$. The index i in $K_{j,i}(z)$ shows the number of integrations of the function $K_j(z)$. Also denote by $(K_{j,i})(z) = K_{j,i}(z)$.

Theorem 1. *Let* $\alpha_0 \neq 0$. *Then:*

(i) Volterra integro-differential equation $\hat{A}u(x) - Ku(x) = 0$ *or*

$$\sum_{i=0}^{n} \alpha_i u^{(n-i)}(x) - \sum_{j=0}^{n} \int_{\alpha}^{x} K_j(x-t) u^{(j)}(t)\,dt = 0 \tag{2}$$

is reduces to Volterra integral equation

$$u(x) - \int_a^x \widehat{K}(x-t) u(t)\,dt = 0, \tag{3}$$

where

$$\widehat{K}(x-t)=\frac{1}{\alpha_0}\left(\mathcal{K}_{0,n}+\mathcal{K}_{1,n-1}+\mathcal{K}_{2,n-2}+\mathcal{K}_{3,n-3}+\ldots+\mathcal{K}_{n,0}\right)(x-t)-$$
$$-\frac{1}{\alpha_0}\left[\alpha_1+\alpha_2(x-t)+\ldots+\frac{\alpha_n}{(n-1)!}(x-t)^{n-1}\right],$$

and has a unique zero solution.

(ii) The operator $\widehat{A}-K$ is closed and continuously invertible.

(iii) If the functions u(x), g(x), f(x) are of exponential order α then the nonhomogenious equation $\widehat{A}u(x)-Ku(x)=f(x)$ for each f(x) has a unique solution

$$u(x)=\mathcal{L}^{-1}\left[\frac{\mathbb{F}(s)}{\alpha_0 s^n+\alpha_1 s+\ldots+\alpha_n-\mathbb{K}(s)}\right],\ x\in[a,+\infty]. \tag{4}$$

Where $\mathbb{F}(s)=\mathcal{L}\left[f(x)\right],\ \mathbb{K}(s)=\mathcal{L}\left[K(x)\right]\neq\alpha_0 s^n+\alpha_1 s+\ldots.+\alpha_n.$

Proof. (i), (ii). First for $j=0,1,\ldots,n$, $i=0,1,\ldots,n-1$ we prove two formulas

$$\int_a^x\int_a^y K_j(y-t)u^{(j)}(t)\,dtdy=\int_a^x K_j(x-t)u^{(j-1)}(t)\,dt, \tag{5}$$

$$\int_a^x\int_a^y K_{j,i}(y-t)u(t)\,dtdy=\int_a^x K_{j,i+1}(x-t)u(t)\,dt, \tag{6}$$

It is easy to verify that

$$\int_t^x \mathcal{K}_{j,i}(y-t)\,dy=\mathcal{K}_{j,i+1}(x-t),\ j=0,1,\ldots,n,\ i=0,1,\ldots,n-1. \tag{7}$$

Further by using Fubbini theorem and integrating by parts we obtain

$$\int_a^x\int_a^y K_j(y-t)u^{(j)}(t)\,dtdy=\int_a^x u^{(j)}(t)\left[\int_t^x K_j(y-t)\,dy\right]dt=$$
$$=\left[u^{(j-1)}(t)\int_t^x K_j(y-t)\,dy\right]_{t=a}^{t=x}-\int_a^x u^{(j-1)}(t)\left(K_{j,1}(x-t)\right)'_t\,dt.$$

Now taking into account (7) we get

$$\int_a^x\int_a^y \mathcal{K}_j(y-t)u^{(j)}(t)\,dtdy = -\int_a^x u^{(j-1)}(t)\left(\mathcal{K}_{j,1}(x-t)\right)'_t dt = \int_a^x u^{(j-1)}(t)K_j(x-t)\,dt.$$

So we proved (5). Further by using Fubbini theorem and (7) we have

$$\int_a^x\int_a^y \mathcal{K}_{j,i}(y-t)u(t)\,dtdy = \int_a^x \mathcal{K}_{j,j+1}(x-t)u(t)\,dt.$$

So we proved (6). Now we will prove that $\widehat{A}-K$ is injective. Rewrite (2) as

$$\alpha_0 u^{(n)}(x)+\alpha_1 u^{(n-1)}(x)+\alpha_2 u^{(n-2)}(x)+\ldots+\alpha_n u(x)=$$
$$=\int_\alpha^x K_0(x-t)u(t)\,dt+\int_a^x K_1(x-t)u'(t)\,dt+$$
$$+\int_a^x K_2(x-t)u''(t)\,dt\ldots+\int_a^x K_n(x-t)u^{(n)}(t)\,dt.$$

Integrating with respect to x both sides of the above equation and using the initial conditions and (5), (6) obtain

$$\alpha_0 u^{(n-1)}(x)+\alpha_1 u^{(n-2)}(x)+\alpha_2 u^{(n-3)}(x)+\ldots+\alpha_n\int_\alpha^x u(t)\,dt=$$
$$=\int_\alpha^x (\mathcal{K}_{0,1}+\mathcal{K}_{1,0})(x-t)u(t)\,dt+$$
$$+\int_a^x K_2(x-t)u'(t)\,dt+\ldots+\int_a^x K_n(x-t)u^{(n-1)}(t)\,dt.$$

The second integration gives

$$\alpha_0 u^{(n-2)}(x)+\alpha_1 u^{(n-3)}(x)+\alpha_2 u^{(n-4)}(x)+\ldots+\alpha_n\int_\alpha^x (x-t)u(t)\,dt=$$
$$=\int_\alpha^x (\mathcal{K}_{0,2}+\mathcal{K}_{1,1}+\mathcal{K}_{2,0})(x-t)u(t)\,dt+\ldots+\int_a^x K_n(x-t)u^{(n-2)}(t)\,dt.$$

By the third integration we get

$$\alpha_0 u^{(n-3)}(x)+\alpha_1 u^{(n-4)}(x)+\alpha_2 u^{(n-5)}(x)+\ldots+\frac{1}{2}\alpha_n\int_\alpha^x (x-t)^2 u(t)\,dt=$$

$$=\int_\alpha^x(\mathcal{K}_{0,3}+\mathcal{K}_{1,2}+\mathcal{K}_{2,1}+\mathcal{K}_{3,0})(x-t)u(t)\,dt+\ldots+\int_a^x K_n(x-t)u^{(n-3)}(t)\,dt.$$

By n-th integration obtain

$$\alpha_0 u(x)+\int_\alpha^x\left[\alpha_1+\alpha_2(x-t)+\ldots+\frac{\alpha_n}{(n-1)!}(x-t)^{n-1}\right]u(t)\,dt=$$

$$=\int_\alpha^x\left(\mathcal{K}_{0,n}+\mathcal{K}_{1,n-1}+\mathcal{K}_{2,n-2}+\mathcal{K}_{3,n-3}+\ldots+\mathcal{K}_{n,0}\right)(x-t)u(t)\,dt,$$

or (2). Since $\widehat{K}(z)$ is a continuous function, the equation (3) has a unique zero solution and so (2) has also a unique zero solution. Then the operator $\widehat{A}-K$ is invertible. The closedness of the operator $\widehat{A}$ is proved in [24]. The operator K is bounded as compact operator. Remind that $D\left(\widehat{A}\right)=C_0^n[a,b]$ and $D(K)=C^n[a,b]$. So $D\left(\widehat{A}\right)\subset D(K)$. We will show that $\widehat{A}-K$ is closed. Let $u_n \in D\left(\widehat{A}\right)$ and $u_n \xrightarrow{C^n} u_0$, $\left(\widehat{A}-K\right)u_n \xrightarrow{C} \upsilon$. Then $u_0\in C^n[a,b]$ and $\widehat{A}u_n \xrightarrow{C} \upsilon+Ku_0$. The last relation follows from the boundedness of K. From $u_n \xrightarrow{C^n} u_0$, $\widehat{A}u_n \xrightarrow{C} \upsilon+Ku_0$ since $\widehat{A}$ is closed, follows that $u_0 \in D(\widehat{A})$ and $\widehat{A}u_0 = \upsilon + Ku_0$. Then $\left(\widehat{A}-K\right)u_0=\upsilon$. So the operator $\widehat{A}-K$ is closed, which implies the closedness of $\left(\widehat{A}-K\right)^{-1}$. Now, since $R(\widehat{A}-K)=C[a,b]$, by Closed Graph Theorem, the operator $(A-K)^{-1}$ is bounded and the operator $\left(\widehat{A}-K\right)^{-1}$ is continuously invertible.

(iii) By using Laplace transform on both sides of $\widehat{A}u(x)-Ku(x)=f(x)$ and then by inverse Laplace transform we obtain (4). The theorem is proved.

The next theorem is useful for solving of linear Volterra – Fredholm or Volterra loaded integro-differential equations with initial boundary conditions.

Theorem 2. *Let the operator* $B : C_0^n[a,b] \to C[a,b]$ *be defined by the equality*

$$Bu(x) = \widehat{A}u(x) - Ku(x) - g\Phi(u)(x) = f(x). \tag{8}$$

Then:

(i) R(K) $\subset$ *$C_0[a,b]$, the operator K is compact and R(*$\widehat{A} - K$*) = C[a,b].*

(ii) The operator B is invertible if and only if detW $\neq$ *0, where*

$$W = \left[I_m - \Phi\left(\left(\widehat{A} - K \right)^{-1} g \right) \right] \tag{9}$$

Proof. (i) Let $z(x) = Ku(x) = \sum_{j=0}^{n} \int_a^x K_j(x-t) u^{(j)}(t)\,dt.$ Then $z(a) = 0$ and $R(K) \subset C_0[a, b]$. Denote by $K_j u(x) = \int_a^x K_j(x-t) u^{(j)}(t)\,dt.$

Then $Ku(x) = \sum_{j=0}^{n} K_j u(x).$ We will show that the operators $K_j : C_0^n[a,b] \to C[a,b], j = 0,1,\ldots,n$ are compact. Note that for x_1, $x_2 \in [a, b]$ hold

$$\left| K_j u(x_1) - K_j u(x_2) \right| = \left| \int_a^{x_1} K_j(x_1 - t) u^{(j)}(t)\,dt - \int_a^{x_2} K_j(x_2 - t) u^{(j)}(t)\,dt \right| \le$$

$$\le \left| \int_a^{x_1} \left[K_j(x_1 - t) - K_j(x_2 - t) \right] u^{(j)}(t)\,dt \right| + \left| \int_{x_1}^{x_2} K_j(x_2 - t) u^{(j)}(t)\,dt \right| \le$$

$$\le \left\| u^{(j)}(t) \right\|_C ((b-a) \max_t \left| K_j(x_1 - t) - K_j(x_2 - t) \right| + |x_1 - x_2| \max_t \left| K_j(x_2 - t) \right| \le$$

$$\le \left\| u(t) \right\|_{C^n} \left((b-a) \max_t \left| K_j(x_1 - t) - K_j(x_2 - t) \right| + |x_1 - x_2| \max_t \left| K_j(x_2 - t) \right| \right).$$

Since $K_j(x - t)$ is continuous, it is uniformly continuous and $|K_j u(x_1) - K_j u(x_2)|$ can be made arbitrarily small by taking $|x_1 - x_2|$ small. How small $|x_1 - x_2|$ should be depends only on $\left\| u(t) \right\|_{C^n}$. In other words, the $K_j u$ are equicontinuous for a bounded set of u. Sinse $\{K_j u\}$ is equibounded for a bounded set $\{u\}$ for a similar reason, by Ascoli – Arzeia criterion follows that $\{K_j u_i\}$ contains a uniformly convergent subsequence if $\{u_i\}$ is bounded. Since this means that $\{K_j u_i\}$ contains a Cauchy subsequence in $C[a, b]$, then K_j, $j = 0, 1, ..., n$ is compact. From the last follows that $K = \sum_{j=0}^{n} K_j$ is a compact operator as a finite sum of compact operators. Now we find $R(\widehat{A}-K)$. Let $(\widehat{A}-K)u=y$, $y\in C[a, b]$. This equation is equivalent to $(I-\widehat{A}^{-1}K)u=\widehat{A}^{-1}y$. The operator $\widehat{A}^{-1}K$ is compact, because K is compact and $\widehat{A}^{-1}$ is bounded. By second Fredholm Theorem $R\left(I-\widehat{A}^{-1}K\right)=$ $=\left\{y\in C[a,b]:\psi\left(\widehat{A}^{-1}y\right)=0\right\}$, where $\psi \in \ker(I-K^*\widehat{A}^{-1*})$. By the first Fredholm Theorem $\dim\ker(I-K^*\widehat{A}^{-1*}) = \dim\ker(I-\widehat{A}^{-1}K)$. But $\ker(I-\widehat{A}^{-1}K) = \ker(\widehat{A}-K) = \{0\}$, since $\widehat{A}-K$ by Theorem 1 is invertible. So $\ker(I-K^*\widehat{A}^{-1*}) = \{0\}$ and $R(I-\widehat{A}^{-1}K)=C_0^n[a,b]$. Then $R(\widehat{A}-K) = C[a, b]$.

(ii) Since the operator B is linear, it is sufficiently to prove that B is injective.

Let $Bu = 0$ and $\det W \neq 0$. Then

$$Bu = \hat{A}u - Ku - g\Phi(u) = 0. \tag{10}$$

Taking into account that the operator $\widehat{A}-K$ is invertible and $R(\widehat{A}-K) = C[a, b]$ we can use the inverse operator $(\widehat{A}-K)^{-1}$ on both sides of Equation (10) and get

$$u = (\widehat{A}-K)^{-1}g\Phi(u). \tag{11}$$

Applying the vector Φ on (11) and using the linearity of $\Phi_1, ..., \Phi_m$, we arrive at the equations

$$\Phi(u)=\Phi\left(\left(\widehat{A}-K\right)^{-1}g\right)\Phi(u),$$

$$\left[I_m-\Phi\left(\left(\widehat{A}-K\right)^{-1}g\right)\right]\Phi(u)=0,$$

$$W\Phi(u)=0. \tag{12}$$

From the last equation, since $\det W \neq 0$, follows that $\Phi(u) = 0$. Substituting $\Phi(u) = 0$ into (11) we obtain $u = 0$. So $\ker B = \{0\}$ and the operator B is invertible.

Conversely. Let $\det W = 0$. Then there exists a nonzero vector $\mathbf{c} = col(c_1, ..., c_m)$ such that $W\mathrm{c} = 0$. Consider the element $u_0 = (\widehat{A} - K)^{-1} g\mathbf{c}$. Observe that $u_0 \neq 0$ since $g_1, ..., g_m$ is a linearly independent set in $C[a, b]$. Then $\mathrm{B}u_0 = (\widehat{A} - K)u_0 - g\Phi(u_0) = g\mathbf{c} - g\Phi((\widehat{A} - K)^{-1}g)\mathbf{c} = g[I_m - \Phi((\widehat{A} - K)^{-1}g)]\mathbf{c} = gW\mathbf{c} = \mathbf{0}$.

Consequently, $u_0 \in \ker B$ and B is not injective. Hence the operator B is invertible if and only if $\det W \neq 0$. The theorem is proved. □

Corollary. *The equation $Bu = f$ for each $f \in C[a, b]$ has a unique solution*

$$u = \left(\widehat{A} - K\right)^{-1} f + \left(\widehat{A} - K\right) gW^{-1}\Phi\left(\left(\widehat{A} - K\right)^{-1} f\right) \tag{13}$$

if and only if $\det W \neq 0$. The operator B is continuously invertible on $C[a, b]$ and

$$B^{-1} f = \left(\widehat{A} - K\right)^{-1} f + \left(\widehat{A} - K\right)^{-1} gW^{-1}\Phi^{-1}\left(\left(\widehat{A} - K\right)^{-1} f\right). \tag{14}$$

Proof. Acting by the inverse operator $(\widehat{A} - K)^{-1}$ on both sides of Equation (8) we get

$$u = (\widehat{A} - K)^{-1} f + (\widehat{A} - K)^{-1} g\Phi(u). \tag{15}$$

Applying the vector Φ on (15) and using the linearity of $\Phi_1, ..., \Phi_m$, we arrive at the equations

$$\Phi(u) = \Phi\left(\left(\widehat{A} - K\right)^{-1} f\right) + \Phi\left(\left(\widehat{A} - K\right)^{-1} g\right)\Phi(u),$$

$$\left[I_m - \Phi\left(\left(\widehat{A} - K\right)^{-1} g\right)\right]\Phi(u) = \Phi\left(\left(\widehat{A} - K\right)^{-1} f\right),$$

$$\Phi(u) = W^{-1}\Phi\left(\left(\widehat{A} - K\right)^{-1} f\right). \tag{16}$$

Substituting (16) into (15) we obtain the exact solution (13) of (8). The boundedness of the operator $(\widehat{A} - K)^{-1}$ on $C[a, b]$ and the boundedness of the functionals $\Phi_1, ..., \Phi_m$ on $C^{n-1}[a, b]$ imply the boundedness of the operator B^{-1} on $C[a, b]$.

In the next examples we assume that a function $u(x)$ is of exponential order α, $0 \le x < \infty$.

Example 1. The next linear Volterra – Fredholm integro-differential equation

$$u''(x)+2\int_0^x \sin 2(x-t)u(t)\,dt-\cos\left(\sqrt{2}x\right)\int_0^\pi u(t)\,dt=\sin\left(\sqrt{2}x\right), \quad (17)$$

$$u(0)=0, u'(0)=0,\ u(x)\in C_0^2[0,\pi],$$

has the unique exact solution

$$\begin{aligned} u(x) &= \frac{\left(7-2x^2\right)}{16}\sin\left(\sqrt{2}x\right)-\frac{7}{16}\sqrt{2}x\cos\left(\sqrt{2}x\right)+ \\ &+\frac{10x\sin\left(\sqrt{2}x\right)-2\sqrt{2}x^2\cos\left(\sqrt{2}x\right)}{7\sqrt{2}\pi\cos\left(\sqrt{2}\pi\right)+\left(2\pi^2-7\right)\sin\left(\sqrt{2}\pi\right)+16\sqrt{2}}\times \\ &\times\left[\left(\frac{\sqrt{2}\pi^2}{16}-\frac{\sqrt{2}}{2}\right)\cos\left(\sqrt{2}\pi\right)-\frac{9\pi}{16}\sin\left(\sqrt{2}\pi\right)+\frac{\sqrt{2}}{2}\right]. \end{aligned} \quad (18)$$

Proof. If we compare the equation (17) with (8) it is natural to take $n = 2$, $m = 1$, $\widehat{A}u(x)=u''(x)$, $D(\widehat{A})=\{u(x)\in C^2[0,\pi]:u(0)=0,\ u'(0)=0\}$, $Ku(x)=-2\int_0^x \sin 2(x-t)u(t)\,dt$, $\Phi(u)=\int_0^\pi u(t)\,dt$, $Bu=\widehat{A}u-Ku-g\Phi(u)$, $Ku-g\Phi(u)$, $D(B)=D(\widehat{A})$, $g(x)=\cos\left(\sqrt{2}x\right)$, $f(x)=\sin\left(\sqrt{2}x\right)$. Note that $\left|\Phi(u)\right|\le\int_0^\pi\left|u(t)\right|dt\le\pi\max_{t\in[0,\pi]}\left|u(t)\right|=\pi u_C$. This means that $\Phi\in C[0,\pi]^*$. Then $\Phi\in C^1[0,\pi]^*$. We can use Theorem 2. Let $\mathcal{L}$, $\mathcal{L}^{-1}$ be the operators of the direct and inverse Laplace transforms, respectively. Denote by $\mathcal{L}[u(x)] = U(s)$ and $\mathcal{L}[y(x)] = Y(s)$. The functions $\sin\left(\sqrt{2}x\right)$, $\cos\left(\sqrt{2}x\right)$, $\sin 2x$ are continuous on each closed interval $[0, b]$, $b < \infty$ and of exponential order 0. So we can use Laplace transform. Note that every solution of (17) on $[0, \infty)$ is also the solution of (17) on $[0, \pi]$. From

$$(A-K)u(x)=u''(x)+2\int_0^x \sin 2(x-t)u(t)\,dt = y(x),\ \ u(0)=0,\ \ u'(0)=0,$$

by using the Laplace transform and convolution operator we get

$$s^2U(s)+\frac{4}{s^2+4}U(s)=Y(s) \text{ or } U(s)=\left(\frac{1}{s^2+2}+\frac{2}{\left(s^2+2\right)^2}Y(s)\right),$$

or

$$U(s)=\left(\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{s^2+\left(\sqrt{2}\right)^2}+\frac{\sqrt{2}}{4}\frac{2\left(\sqrt{2}\right)^3}{\left(s^2+\left(\sqrt{2}\right)^2\right)^2}\right)Y(s).$$

Now by using the inverse Laplace transform we obtain

$$u(x)=\left(\frac{3\sqrt{2}}{4}\sin\left(\sqrt{2}x\right)-\frac{1}{2}x\cos\left(\sqrt{2}x\right)\right)y(x)$$

or

$$\left(\widehat{A}-K\right)^{-1}y(x)=\int_0^x\left[\frac{3\sqrt{2}}{4}\sin\left(\sqrt{2}(x-t)\right)-\frac{1}{2}(x-t)\cos\left(\sqrt{2}(x-t)\right)\right]y(t)\,dt.$$

Then for $g(x)=\cos\left(\sqrt{2}x\right)$ and $f(x)=\sin\left(\sqrt{2}x\right)$ we get

$$\left(\widehat{A}-K\right)^{-1}g=$$

$$=\int_0^x\left[\frac{3\sqrt{2}}{4}\sin\left(\sqrt{2}(x-t)\right)-\frac{1}{2}(x-t)\cos\left(\sqrt{2}(x-t)\right)\right]\cos\left(\sqrt{2}x\right)dt=$$

$$=\frac{5\sqrt{2}}{16}x\sin\left(\sqrt{2}x\right)-\frac{1}{8}x^2\cos\left(\sqrt{2}x\right),$$

$$\left(\widehat{A}-K\right)^{-1}f=$$

$$=\int_0^x\left[\frac{3\sqrt{2}}{4}\sin\left(\sqrt{2}(x-t)\right)-\frac{1}{2}(x-t)\cos\left(\sqrt{2}(x-t)\right)\right]\sin\left(\sqrt{2}t\right)dt=$$

$$=\frac{1}{16}\left[\left(7-2x^2\right)\sin\left(\sqrt{2}x\right)-7\sqrt{2}x\cos\left(\sqrt{2}x\right)\right].$$

Further we compute

$$\Phi\left(\left(\widehat{A}-K\right)^{-1} g\right)=\frac{1}{32}\left[\sqrt{2}\left(7-2\pi^2\right)\sin\left(\sqrt{2}\pi\right)-14\pi\cos\left(\sqrt{2}\pi\right)\right],$$

$$\Phi\left(\left(\widehat{A}-K\right)^{-1} f\right)=\left(\frac{\sqrt{2}\pi^2}{16}-\frac{\sqrt{2}}{2}\right)\cos\left(\sqrt{2}\pi\right)-\frac{9\pi}{16}\sin\left(\sqrt{2}\pi\right)+\frac{\sqrt{2}}{2},$$

$$\left[I_n-\Phi\left(\left(\widehat{A}-K\right)^{-1} g\right)\right]^{-1}=\frac{16\sqrt{2}}{7\sqrt{2}\pi\cos\left(\sqrt{2}\pi\right)+\left(2\pi^2-7\right)\sin\left(\sqrt{2}\pi\right)+16\sqrt{2}}.$$

Substituting these values into (13) we obtain (18). ▲

Example 2. The next linear Volterra loaded integro-differential equation

$$u''(x)+3\int_0^x \cos(x-t)u'(t)\,dt-u(\pi)\sin x=(2\pi-8)\sin x, \tag{19}$$

$$u(0) = 0,\ u'(0) = 0, \qquad u(x) \in C_0^2[0, \pi],$$

has the unique exact solution

$$u(x) = \sin 2x - 2x. \tag{20}$$

Proof. If we compare the equation (19) with (8) it is natural to take $n = 2$, $m = 1$, $\widehat{A}\,u(x) = u''(x)$, $D(\widehat{A}) = \{u(x) \in C^2[0, \pi]: u(0) = 0,\ u'(0) = 0\}$,

$Ku(x) = -3\int_0^x \cos(x-t)u'(t)\,dt$, $\Phi(u) = u(\pi)$, $Bu = \widehat{A}\,u - Ku - g\Phi(u)$, $D(B) = D(\widehat{A})$, $g(x) = \sin x$, $f(x) = (2\pi - 8)\sin x$. Note that $|\Phi(u)| = |u(\pi)| \le \|u\|_C$. This means that $\Phi \in C[0, \pi]^*$. We can use Theorem 2. Let $\mathcal{L}$, $\mathcal{L}^{-1}$ be the operators of the direct and inverse Laplace transforms, respectively. Denote by $\mathcal{L}[u(x)] = U(s)$ and $\mathcal{L}[y(x)] = Y(s)$. The functions $\sin x$, $\cos x$ are continuous on each closed interval $[0, b]$, $b < \infty$ and of exponential order 0. So we can use Laplace transform. Note that every solution of (19) on $[0, \infty)$ is also the solution of (19) on $[0, \pi]$. From

$$\left(\widehat{A}-K\right)u(x)=u''(x)+3\int_0^x \cos(x-t)u'(t)\,dt=y(x),\ u(0)=0,\ u'(0)=0,$$

by using the Laplace transform and convolution operator we get

$$s^2U(s)+\frac{3s^2}{s^2+1}U(s)=Y(s) \text{ or } U(s)=\frac{s^2+1}{s^2\left(s^2+4\right)}Y(s),$$

or

$$U(s)=\left(\frac{3}{4}\frac{1}{s^2+4}+\frac{1}{4}\frac{1}{s^4}\right)Y(s).$$

Now by using the inverse Laplace transform we obtain

$$u(x)=\left(\frac{3}{8}\sin 2x+\frac{1}{4}x\right)y(x)$$

or

$$\left(\widehat{A}-K\right)^{-1}y(x)=\int_0^x\left[\frac{3}{8}\sin 2(x-t)+\frac{1}{4}(x-t)\right]y(t)\,dt.$$

Then for $g(x)=\sin x$ and $f(x)=(2\pi-8)\sin x$ we get

$$\left(\widehat{A}-K\right)^{-1}g=\int_0^x\left[\frac{3}{8}\sin 2(x-t)+\frac{1}{4}(x-t)\right]\sin t\,dt=\frac{x}{4}-\frac{\sin 2x}{8}$$

and

$$\left(\widehat{A}-K\right)^{-1}f=\int_0^x\left[\frac{3}{8}\sin 2(x-t)+\frac{1}{4}(x-t)\right](2\pi-8)\sin t\,dt=$$

$$=(2\pi-8)\left(\frac{x}{4}-\frac{\sin 2x}{8}\right).$$

Further we compute

$$\Phi\left(\left(\widehat{A}-K\right)^{-1}g\right)=\left(\left(\widehat{A}-K\right)^{-1}g\right)(\pi)=\pi/4,$$

$$\Phi\left(\left(\widehat{A}-K\right)^{-1}f\right)=\left(\left(\widehat{A}-K\right)^{-1}f\right)(\pi)=(2\pi-8)\pi/4,$$

$$\left[I_n-\Phi\left(\left(\widehat{A}-K\right)^{-1}g\right)\right]^{-1}=\frac{4}{4-\pi}.$$

Substituting these values into (13) we obtain (20).

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