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LOADED DIFFERENTIAL AND FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL INTEGRAL BOUNDARY CONDITIONS

A direct method for the exact solution of loaded differential or Fredholm Integro-Differential Equations (IDEs) with nonlocal integral boundary conditions is proposed. We consider the abstract operator equations of the form $Bu = Au - g\Psi(u) = f$ with abstract nonlocal boundary conditions. In this paper we investigate the correctness of the equation $Bu = f$ and its exact solution in closed form.

Keywords: Loaded differential equations, integro-differential equations, nonlocal integral boundary conditions, injective and correct operators, exact solutions.

Introduction

In recent years the theory of loaded functional, differential and integro-differential equations (IDEs) has been advanced. These equations describe problems in optimal control, regulation of the layer of soil water and ground moisture, underground fluid and gas dynamics, mathematical biology, economics, ecology, and pure mathematics [1–7]. An equation is called loaded equation if it contains the solution function on a manifold with dimension less than the dimension of domain of this function [2]. For example, an ordinary loaded differential equation is represented by

$$dy/dx = f(x, y) + \psi(y(x_j)), x \in [0, 1], x_j \in [0, 1],$$

where x_j are fixed points. Boundary value problems (BVPs) for differential and IDEs with nonlocal boundary conditions arise in various fields of mechanics, physics, biology, biotechnology, chemical engineering, medical science, finance and others (see [8–15]). Fredholm IDEs with nonlocal integral boundary conditions and ordinary differential operators, probably, first were considered by J.D. Tamarkin [16]. Nonlocal BVPs for the ordinary differential equations with integral boundary conditions were studied in [17–19]. Nonlocal BVPs for the partial differential equations were investigated in [8, 20–25]. Loaded nonlocal BVPs for ordinary differential equations were considered in [7, 26]. This work is a generalization of the papers [27–32], where integral

boundary conditions have not been considered. The problems of the type $Au - g\Psi(u) = f$, $\Phi(u) = NF(Au)$ arise naturally from A.A. Dezin, R.O. Oinarov extensions of linear operators [33, 34], which are not restrictions of a maximal operator, unlike the classical M.G. Krein, J.Von. Neuman extensions [35, 36] in Hilbert space and M. Otelbaev [37] in Banach space. We investigate the abstract operator equations of the form $Bu = Au - g\Psi(u) = f$ with abstract nonlocal boundary conditions $D(B) = \{u \in D(A) : \Phi(u) = NF(Au)\}$, where $B : X \to Y$, X, Y are Banach spaces, $g \in Y^n$ is a vector, $F \in Y^{n^*}$, $\Psi \in X_A^{n^*}$ are vector functionals, N is a matrix, A is a differential operator with finite-dimensional kernel and $g\Psi(u)$ is a loaded part of $Bu = f$ or a Fredholm integro-differential operator defined on the space with graph norm $X_A = \left(D(A), \|\cdot\|_{X_A} \right)$. If $g = 0$, we have a differential equation with nonlocal integral boundary conditions. The technique presented here allows us to find exact solutions in closed form for loaded Differential or Fredholm Integro-Differential Equations with separable kernels and integral boundary conditions. This technique is simple to use and can be easily incorporated to any Computer Algebra System (CAS). The essential ingredient in our approach is the extension of the main idea in [34]. The paper is organized as follows. In Section 1 we recall some basic terminology and notation about operators. In Section 2 we prove the main general results. Finally, in Section 3 we discuss some examples of loaded differential and IDEs which show the usefulness of our technique.

1. Terminology and notation

Let X be a complex Banach space and X^* its adjoint space, i.e. the set of all complex-valued linear and bounded functionals on X. We denote by $f(x)$ the value of f on x. We write $D(A)$ and $R(A)$ for the domain and the range of the operator A, respectively. An operator A_2 is said to be an *extension* of an operator A_1, or A_1 is said to be a *restriction* of A_2, in symbol $A_1 \subset A_2$, if $D(A_2) \supseteq D(A_1)$ and $A_1 x = A_2 x$, for all $x \in D(A_1)$. An operator $A: X \to Y$ is called closed if for every sequence x_n in $D(A)$ converging to x_0 with $Ax_n \to f_0$, it follows that $x_0 \in D(A)$ and $Ax_0 = f_0$. A closed operator $A_0 : X \to Y$ is called *minimal* if $R(A_0) \neq Y$ and the inverse A_0^{-1} exists on $R(A_0)$ and is continuous. A closed operator A is called *maximal* if $R(A) = Y$ and ker $A \neq \{0\}$. An operator $\hat{A}$ is called *correct* if $R(\hat{A}) = Y$ and the inverse $\hat{A}^{-1}$ exists and is con-

tinuous. The problem $\hat{A}x = f$ is said to be *correct* when the operator $\hat{A}$ is correct. If $\Psi_i \in X^*$, $i = 1, ..., m$, then we denote by $\Psi = \text{col}(\Psi_1, ..., \Psi_m)$ and $\Psi(x) = \text{col}(\Psi_1(x), ..., \Psi_m(x))$. Let $q = (q_1, ..., q_m)$ be a vector of X^m. We will denote by $\Psi(q)$ the $m \times m$ matrix whose i, j-th entry is the value of functional Ψ_i on element q_j and by I_m and 0_m the identity $m \times m$ and the zero $m \times m$ matrices, respectively. By $\vec{0}$ we will denote the zero column vector.

2. Loaded differential and Fredholm IDE
with nonlocal integral boundary conditions

Lemma 2.1. *Let the operator A: C[0, 1] → C[0, 1] be defined by $Au(t) = u'(t)$, $D(A) = C^1[0, 1]$, $\|u(t)\| = \max\limits_{t \in [0,1]} |u(t)|$. Then:*

(i) The operator A is closed and A^2: C[0, 1] → C[0, 1] is determined by $A^2u(t) = u''(t)$, $D(A^2) = C^2[0, 1]$ and closed too.
(ii) For all $u(t) \in C^2[0, 1]$ the next inequality holds

$$\|u'(t)\| \leq \|u''(t)\| + 2\|u(t)\|. \tag{1}$$

Proof. (i)-(ii) Let x_1, $x_2 \in [0, 1]$. Then from the equality $u'(x_2) - u'(x_1) = \int\limits_{x_1}^{x_2} u''(x) dx$ we obtain

$$u'(x_2) - u'(x_1) \leq \|u''(x)\|,$$

$$\int\limits_0^1 u'(x_2) dx_1 - \int\limits_0^1 u'(x_1) dx_1 \leq \|u''(x)\|,$$

$$u'(x_2) \leq \|u''(x)\| + u(1) - u(0),$$

$$\max\limits_{x_2 \in [0,1]} |u'(x_2)| \leq u''(x) + 2\|u(x)\|.$$

From the last relation follows (1). Now we will prove the closedness of A^2. Let $u_n(t) \in D(A^2)$, $u_n(t) \to u(t)$, $n \to \infty$ and $A^2u_n(t) = u_n''(t) \to y(t)$, $n \to \infty$. Put $y_n(t) = u_n''(t)$. Then

$$\int\limits_0^t y_n(x) dx = \int\limits_0^t u_n''(x) dx = u_n''(t) - u_n''(0) \tag{2}$$

From (1) we get

$$\|u_n'(t) - u_m'(t)\| \leq \|u_n''(t) - u_m''(t)\| + 2\|u_n(t) - u_m(t)\|.$$

From this inequality, because of $u_n(t)$ and $u_n''(t)$ are fundamental sequences in $C[0, 1]$, it follows that $u_n'(t)$ is also a fundamental sequence in Banach space $C[0, 1]$. Then there exists a function $v(t) \in C[0, 1]$ such that $u_n'(t) \to v(t)$, $n \to \infty$. The last implies $u_n'(0) \to v(0)$, $n \to \infty$. For $n \to \infty$ from (2) we get

$$\int_0^t y(t)\,dx = v(t) - v(0).$$

It is evident that $v(t) \in C^1[0, 1]$ and $v'(t) = y(t)$. From $u_n(t) \to u(t)$, $u_n'(t) \to v(t)$, $n \to \infty$, since the operator A is closed, we get $u(t) \in C^1[0, 1]$ and $u'(t) = v(t)$. But $v(t) \in C^1[0, 1]$ and $v'(t) = y(t)$. Hence $u(t) \in C^2[0, 1]$, $u''(t) = y(t)$ and A^2 is a closed operator.

Lemma 2.2. Let A, B, C, D, E be n×n matrices and $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det \begin{pmatrix} A \pm EC & B \pm ED \\ C & D \end{pmatrix}. \tag{3}$$

Proof. Let $F = \begin{pmatrix} I_n & E \\ 0_n & I_n \end{pmatrix}$. In this case $|F| = |F^{-1}| = 1$. So $|F\,G| = |F|\,|G| =$ $= |G|$ and $|F^{-1}G| = |F^{-1}|\,|G| = |G|$. From the last relations we get (3).

The next theorem easy follows from Theorem 3 [32].

Theorem 2.3. *Let $u(t) \in W_2^l(a, b)$, $h_j(t) \in L_2(a, b)$, $j = 0, 1, ..., l$, $(a_i, b_i) \subseteq (a, b)$, $i = 1, ..., n$ and the functional*

$$\Psi_i(u) = \int_{a_i}^{b_i} \left[\sum_{j=0}^l u^{(j)}(t)\overline{h_j(t)} \right] dt. \tag{4}$$

Then $\Psi_i(u)$ is linear and bounded on $W_2^l(a, b)$ and hence $\Psi_i \in [C^l[a, b]]^*$, $i = 1, ..., n$.

We generalize Theorem 1 [32] without using the linear independence of components of the vector Ψ.

Theorem 2.4. *Let X, Y and Z be complex Banach spaces and $\hat{A} : X \to Y$ a linear correct operator with $D(\hat{A}) \subset Z \subseteq X$. Further let the components of the vector $g = (g_1, \ldots, g_n)$, be linearly independent in Y and the vector $\Psi = col(\psi_1, \ldots, \psi_n) \in [Z^*]^n$. Then:*

(i) The operator B defined by

$$Bu = \hat{A}u - g\Psi(u) = f, D(B) = D(\hat{A}), f \in Y. \tag{5}$$

is injective if and only if

$$\det W = \det[I_n - \Psi(\hat{A}^{-1}g)] \neq 0, \tag{6}$$

(ii) If B is injective, then for any f $\in$ Y the unique solution of (5) is given by

$$u = B^{-1}f = \hat{A}^{-1}f + (\hat{A}^{-1}g)[I_n - \Psi(\hat{A}^{-1}g)]^{-1}\Psi(\hat{A}^{-1}f). \tag{7}$$

Proof. (i). Let det $W \neq 0$. Since f, $g_i \in R(\hat{A})$, $i = 1, ..., m,$ by using (5) we have

$$u - \hat{A}^{-1}g\Psi(u) = \hat{A}^{-1}f, \tag{8}$$
$$\Psi(u) - \Psi(\hat{A}^{-1}g)\Psi(u) = \Psi(\hat{A}^{-1}f),$$
$$[I_m - \Psi(\hat{A}^{-1}g)]\Psi(u) = \Psi(\hat{A}^{-1}f), \tag{9}$$
$$\Psi(u) = [I_n - \Psi(\hat{A}^{-1}g)]^{-1}\Psi(\hat{A}^{-1}f) \text{ for all } f \in R(\hat{A}). \tag{10}$$

Since $\Psi(u)$ is defined uniquely by (9) for all $f \in R(\hat{A}) = Y$, then by substituting (10) into (8), we obtain the unique solution (7) of the problem (5) for all $f \in Y$. So B is an injective operator and $R(\hat{A}) = Y$. Conversely. Let det $W = 0$. Then there exists a vector $\vec{c} = col(c_1, ..., c_n) \neq \vec{0}$ such that $W\vec{c} = 0$. For $u_0 = \hat{A}^{-1}g\vec{c} \neq 0$ follows that

$$Bu_0 = \hat{A}u_0 - g\Psi(u_0) = g\vec{c} - g\Psi(\hat{A}^{-1}g)\,\vec{c} =$$
$$= g[I_n - \Psi(\hat{A}^{-1}g)]\,\vec{c} = gW\vec{c} = g\vec{0} = 0.$$

Hence $u_0 = \hat{A}^{-1}g\vec{c} \in \ker B \neq \{0\}$. Note that $u_0 \neq \vec{0}$, because of $g_1, ..., g_n$ are linearly independent and the operator $\hat{A}$ is correct. So B is not injective. So the statement (i) holds.

(ii). We have proved in (i) that in the case det $W \neq 0$ Equations (8)–(10) hold. Substituting (10) into (8), we obtain the unique solution (7) of the problem (5) for all $f \in Y$. This means that $R(B) = Y$. The boundedness of B^{-1} on Y follows from the boundedness of the operator $\hat{A}^{-1}$ and $\Psi_1, ..., \Psi_n$. Hence B is correct.

Remark 2.5. The linear independence of $g_1, ..., g_n$ has been used only to prove the necessary condition for injectivity of the operator B.

Now we generalize this theorem on the case of perturbed boundary conditions.

Let X, Y be Banach spaces, $A : X \overset{on}{\rightarrow} Y$ be a linear closed operator. We remind that for A, the *graph norm* associated with A, is defined by:

$$||x||_{XA} = ||x||_X + ||Ax||_Y \ \ \forall x \in D(A) \tag{11}$$

and that the set

$$X_A = (D(A), ||\cdot||_{XA}) \tag{12}$$

is a Banach space. Note that usually in the previous theorem $Z = X_A$.

Theorem 2.6. *Let X, Y be complex Banach spaces, $A : X \rightarrow Y$ a maximal linear operator with finite dimensional kernel $z = (z_1, ..., z_m)$ which is a basis of $\ker A$ and $\widehat{A}$ a correct restriction of A defined by*

$$\widehat{A} \subset A, D(\widehat{A}) = \{u \in D(A) : \Phi(u) = 0\}. \tag{13}$$

Suppose also that the components of the vector functionals $\Phi =$ $= col(\Phi_1, ..., \Phi_m)$, $\Psi = col(\Psi_1, ..., \Psi_m)$, $F = col(F_1, ..., F_m)$, belong to X^*_A, X^*_A, Y^*, respectively, where $\Phi_1, ..., \Phi_m$ biorthogonal to $z_1, ..., z_m$ and that the vector $g = (g_1, ..., g_m) \in Y^m$ and N is an $m \times m$ matrix, $g_1, ..., g_m$ linearly independent. Then:

(i) The operator B defined by

$$
\begin{aligned}
Bu &= Au - g\Psi(u) = f, \ f \in Y, \\
D(B) &= \{u \in D(A) : \Phi(u) = NF(Au)\},
\end{aligned} \tag{14}
$$

is injective if and only if

$$det\, L_2 = det[I_m - \Psi(\widehat{A}^{-1}g) - \Psi(z)NF(g)] \neq 0. \tag{15}$$

(ii) If B is injective, then B is correct and for all $f \in Y$ the unique solution of (14) is given by

$$u = B^{-1}f = \widehat{A}^{-1}f + p\Psi(\widehat{A}^{-1}f) + [z + p\Psi(z)]NF(f), \ where \tag{16}$$
$$p = [\widehat{A}^{-1}g + zNF(g)]L^{-1}_2. \tag{17}$$

Proof. (i). Since $z \in [\ker A]^m$, $\Phi(z) = I_m$, the problem (14) is written as

$$
\begin{aligned}
Bu &= A(u - zNF(Au)) - g\Psi(u) = f, f \in Y, \\
D(B) &= \{u \in D(A) : \Phi(u - zNF(Au)) = 0\},
\end{aligned} \tag{18}
$$

Then $u - zNF(Au) \in D(\widehat{A})$ for every $u \in D(B)$ and

$$Bu = \widehat{A}\,(u - zNF(Au)) - g\Psi(u) = f, \tag{19}$$
$$u - zNF(Au) - \widehat{A}^{-1}g\Psi(u) = \widehat{A}^{-1}f, \tag{20}$$

$$\Psi(u) - \Psi(z)NF(Au) - \Psi(\hat{A}^{-1}g)\Psi(u) = \Psi(\hat{A}^{-1}f),$$

$$[I_m - \Psi(\hat{A}^{-1}g)]\Psi(u) - \Psi(z)NF(Au) = \Psi(\hat{A}^{-1}f), \quad (21)$$

$$-F(g)\Psi(u) + F(Au) = F(f). \quad (22)$$

From (21), (22) we obtain

$$\begin{pmatrix} I_m - \Psi(\hat{A}^{-1}) & -\Psi(z)N \\ -F(g) & I_m \end{pmatrix} \begin{pmatrix} \Psi(u) \\ F(Au) \end{pmatrix} = \begin{pmatrix} \Psi(\hat{A}^{-1}f) \\ F(f) \end{pmatrix}. \quad (23)$$

Denoting the matrix from the left by L and using the formula (3), where we take $E = \Psi(z)N$, we obtain

$$\det L = \det \begin{pmatrix} I_m - \Psi(\hat{A}^{-1}g) & -\Psi(z)N \\ -F(g) & I_m \end{pmatrix} =$$

$$= \pm\det\left[I_m - \Psi(\hat{A}^{-1}g) - \Psi(z)NF(g) \right] = \det L_2.$$

Let $\det L_2 \neq 0$ and $u \in \ker B$. Then

$$Bu = Au - g\Psi(u) = 0, \ \Phi(u) = NF(Au), \ \det L \neq 0. \quad (24)$$

Now using Equations (19)–(23), where $f = 0$ we get

$$\begin{pmatrix} I_m - \Psi(\hat{A}^{-1}g) & -\Psi(z)N \\ -F(g) & I_m \end{pmatrix} \begin{pmatrix} \Psi(u) \\ F(Au) \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}. \quad (25)$$

From the above equation, since $\det L = \pm \det L_2 \neq 0$ follows that $\Psi(u) = \vec{0}$, $F(Au) = \vec{0}$. Substituting these values into (24), we obtain $\Phi(u) = \vec{0}$ and $Bu = \hat{A}u = 0$. From $\hat{A}u = 0$, since $\hat{A}$ is correct follows that $u = 0$. This proves that $\ker B = \{0\}$. So B is injective.

Conversely. Let $\det L_2 = 0$. Then there exists a vector $\vec{c} = col(c_1,..., c_m) \neq \vec{0}$ such that $L_2\vec{c} = 0$. Observe that he element $u_0 = \hat{A}^{-1}g\vec{c} + zNF(g)\vec{c} \neq 0$, otherwise the components of g will be linearly dependent, which contradicts the hypothesis that $g_1, ..., g_m$ are linearly independent. Note that $u_0 \in D(B)$, since $\Phi(u_0) = NF(g)\vec{c}$, $F(Au_0) = F(g)\vec{c}$ and so $\Phi(u_0) - NF(Au_0) = NF(g)\vec{c} - NF(g)\vec{c} = \vec{0}$. We will show now that $u_0 \in \ker B$.

$$Bu_0 = Au_0 - g\Psi(u_0) = g\vec{c} - g\Psi(\hat{A}^{-1}g\vec{c} + zNF(g)\,\vec{c}) =$$
$$= g\vec{c} - g\Psi(\hat{A}^{-1}g)\,\vec{c} - g\Psi(z)NF(g)\,\vec{c} =$$
$$= g[I_m - \Psi(\hat{A}^{-1}g) - \Psi(z)NF(g)]\,\vec{c} = gL_2\vec{c} = g\vec{0}.$$

Such $u_0 \in \ker B$. Hence $u_0 \in D(B)$ and $u_0 \in \ker B$. So *ker B $\neq$ {0}* and B is not injective. The statement (i) holds.

(ii) Let $\det L_2 \neq 0$. Then $\det L \neq 0$ and from (23) we obtain

$$\begin{pmatrix} \Psi(u) \\ F(Au) \end{pmatrix} = \begin{pmatrix} I_m - \Psi(\hat{A}^{-1}) & -\Psi(z)N \\ -F(g) & I_m \end{pmatrix}^{-1} \begin{pmatrix} \Psi(\hat{A}^{-1}f) \\ F(f) \end{pmatrix}. \tag{26}$$

We put

$$L^{-1} = \begin{pmatrix} I_m - \Psi(\hat{A}^{-1}g) & -\Psi(z)N \\ -F(g) & I_m \end{pmatrix}^{-1} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

Then

$$[I_m - \Psi(\hat{A}^{-1}g)]K_{11} - \Psi(z)NK_{21} = I_m, \tag{27}$$

$$[I_m - \Psi(\hat{A}^{-1}g)]K_{12} - \Psi(z)NK_{22} = 0_m, \tag{28}$$

$$-F(g)K_{11} + K_{21} = 0_m, \tag{29}$$

$$-F(g)K_{12} + K_{22} = I_m. \tag{30}$$

From (30) we obtain $K_{21} = F(g)K_{11}$ and from (27), (29), we get

$$[I_m - \Psi(\hat{A}^{-1}g)]K_{11} - \Psi(z)NF(g)K_{11} = I_m,$$
$$[I_m - \Psi(\hat{A}^{-1}g) - \Psi(z)NF(g)]K_{11} = I_m,$$
$$K_{11} = L_2^{-1}, \ K_{21} = F(g)\ L_2^{-1}.$$

From (28), since $K_{22} = I_m + F(g)K_{12}$, we have

$$[I_m - \Psi(\hat{A}^{-1}g)]K_{12} - \Psi(z)N[I_m + F(g)K_{12}] = 0_m,$$
$$[I_m - \Psi(\hat{A}^{-1}g) - \Psi(z)NF(g)]K_{12} = \Psi(z)N,$$
$$K_{12} = L_2^{-1}\ \Psi(z)N.$$

The last equation and $K_{22} = I_m + F(g)K_{12}$ implies $K_{22} = I_m + F(g)\,L_2^{-1}\Psi(z)N$. So

$$L^{-1} = \begin{pmatrix} L_2^{-1} & L_2^{-1}\Psi(z)N \\ F(g)L_2^{-1} & I_m + F(g)L_2^{-1}\Psi(z)N \end{pmatrix}.$$

Now we rewrite (26) in the form

$$\begin{pmatrix} \Psi(u) \\ F(Au) \end{pmatrix} = \begin{pmatrix} L_2^{-1} & L_2^{-1}\Psi(z)N \\ F(g)L_2^{-1} & I_m + F(g)L_2^{-1}\Psi(z)N \end{pmatrix} \begin{pmatrix} \Psi(\hat{A}^{-1}f) \\ F(f) \end{pmatrix}. \quad (31)$$

Then

$$\Psi(u) = L_2^{-1}\Psi(\hat{A}^{-1}f) + L_2^{-1}\Psi(z)NF(f),$$

$$F(Au) = F(g)\,L_2^{-1}\Psi(\hat{A}^{-1}f) + [I_m + F(g)\,L_2^{-1}\Psi(z)N]F(f).$$

Substituting these values into (20) we get the solution of the problem (14):

$$u = \hat{A}^{-1}f + zN\{F(g)\,L_2^{-1}\Psi(\hat{A}^{-1}f) + [I_m + F(g)\,L_2^{-1}\Psi(z)N]F(f)\}$$
$$- \hat{A}^{-1}g\,[L_2^{-1}\Psi(\hat{A}^{-1}f) + L_2^{-1}\Psi(z)NF(f)],$$

From the last equation for every $f \in Y$ follows the unique solution (16) of (14). Because f in (16) is arbitrary, we obtain $R(B) = Y$. Since the operator $\hat{A}^{-1}$ and the functionals $F_1, ..., F_m, \Psi_1, ... ,\Psi_m$ are bounded, from (16) follows the boundedness of B^{-1}. Hence, the operator B is correct if and only if (15) holds and the unique solution of (14) is given by (16). The theorem is proved.

From the previous theorem for $g = 0$ follows the next corollary which is useful for solving differential equations with integral boundary conditions.

Corollary 2.7. *Let X, Y be complex Banach spaces, the operators A, $\hat{A}$, the vector $z = (z_1, ..., z_m)$ and vector functionals $\Phi = col(\Phi_1, ..., \Phi_m)$, $\Psi = col(\Psi_1, ..., \Psi_m)$ and the matrix N are defined as in Theorem 2.6. Then:*
(i) The operator B defined by

$$Bu = Au = f, f \in Y, D(B) = \{u \in D(A) : \Phi(u) = NF(Au)\}. \quad (32)$$

is correct and for all $f \in Y$ the unique solution of (32) is given by

$$u = B^{-1}f = \hat{A}^{-1}f + zNF(f). \quad (33)$$

Remark 2.8. In applications we encounter operators $B_1 : X \rightarrow Y$ of the form

$$Au - g_1\Psi_1(u) - ... - g_k\Psi_k(u) = f, k \leq m,$$

$$\Phi_1(u) = v_{11}F_1(Au) + ... + v_{1m}F_m(Au)$$

$$... ... ...$$

$$\Phi_m(u) = v_{m1}F_1(Au) + ... + v_{mm}F_m(Au) \tag{34}$$

where X, Y are Banach spaces, Φ_i, F_i, Ψ_j, $i = 1, ..., m$, $j = 1, ..., k$ are functionals. Then we are interested in knowing *whether the operator B_1 is B type operator defined by*

$$Bu = Au - g\Psi(u) = f, f \in Y,$$
$$D(B) = \{u \in D(A) : \Phi(u) = NF(Au)\}, \tag{35}$$

and, therefore, Theorem 2.6 applies. For this purpose, we proceed as follows:

1.1. We check that $m = k$ and show that the operator A is maximal, namely A is closed with dim ker $A = m$ and $R(A) = Y$,

1.2. We find the basis $z_1, ..., z_m$ of ker A and the Banach space $X_A = (D(A), || \cdot ||_{XA})$, where $||u||_{XA} = ||u||_X + ||Au||_Y$,

1.3. We show that the operator $\hat{A}$ defined by

$$\hat{A}u = Au, D(\hat{A}) = \{u \in D(A) : \Phi(u) = \vec{0}\},$$

is correct and find the inverse operator $\hat{A}^{-1}$,

1.4. We find the vector g, the element f, the $m \times m$ matrix N, the functional vectors $\Phi(u) = col(\Phi_1(u), ..., \Phi_m(u))$,

$\Psi(u) = col(\Psi_1(u), ..., \Psi_m(u))$, $F(Au) = col(F_1(Au), ..., F_m(Au))$ and $F(f) = col(F_1(f), ..., F_m(f))$,

1.5. We check the condition $\Phi(z) = I_m$,

1.6. We check the conditions $F_i \in Y^*$, Φ_i and $\Psi_i \in X^*_A$, $i = 1, ..., m$.

If all these conditions hold true then we apply Theorem 2.6. If one of these steps fails, then B_1 is not identified as B-type operator and, therefore, the theory can not be applied.

To prove that B is an injective and correct operator, we proceed as follows:

2.1. We calculate the vector $\hat{A}^{-1}g = (\hat{A}^{-1}g_1, ..., \hat{A}^{-1}g_m)$,

2.2. We calculate the $m \times m$ matrices

2.2.1. $\Psi\left(\hat{A}^{-1}g\right) = \begin{pmatrix} \Psi_1\left(\hat{A}^{-1}g_1\right) & ... & \Psi_1\left(\hat{A}^{-1}g_m\right) \\ ... & ... & ... \\ \Psi_m\left(\hat{A}^{-1}g_1\right) & ... & \Psi_m\left(\hat{A}^{-1}g_m\right) \end{pmatrix},$

$$2.2.2. \quad \Psi(z) = \begin{pmatrix} \Psi_1(z_1) & \dots & \Psi_1(z_m) \\ \dots & \dots & \dots \\ \Psi_m(z_1) & \dots & \Psi_m(z_m) \end{pmatrix},$$

$$2.2.3. \quad F(g) = \begin{pmatrix} F_1(g_1) & \dots & F_1(g_m) \\ \dots & \dots & \dots \\ F_m(g_1) & \dots & F_m(g_m) \end{pmatrix},$$

2.2.4. $L_2 = I_m - \Psi(\hat{A}^{-1}g) - \Psi(z)NF(g)$.

If the determinant $\det L_2 \neq 0$, then B is an injective and correct operator.

To find the solution of (34), we proceed as follows:

3.1. We calculate the inverse matrix L_2^{-1} and the element $\hat{A}^{-1}f$,

3.2. We calculate the vectors $\Psi(\hat{A}^{-1}f) = col(\Psi_1(\hat{A}^{-1}f), ..., \Psi_m(\hat{A}^{-1}f))$,
$F(f) = col(F_1(f), ..., F_m(f))$,
$p = [\hat{A}^{-1}g + zNF(g)]L_2^{-1}$.

3.3. We find the solution of (34) by

$$u = B^{-1}f = \hat{A}^{-1}f + p\Psi(\hat{A}^{-1}f) + [z + p\Psi(z)]NF(f).$$

4. If $k < m$, then we take as vector g the vector $\tilde{g} = (\tilde{g}_1, ..., \tilde{g}_m)$, where $\tilde{g}_i = g_i, i = 1, ..., k, \tilde{g}_i = 0, i = k + 1, ..., m$. As functional vector $\Psi(u)$ we can take $\hat{\Psi}(u) = (\hat{\Psi}_1(u), ..., \hat{\Psi}_m(u))$ with linearly independent elements, where $\hat{\Psi}_i(u) = \psi_i(u), i = 1, ..., k,$ and $\hat{\Psi}_i(u), i = k+1, ..., m$ are arbitrary functionals.

3. Examples

Example 3.1. The next problem with loaded differential equation and nonlocal integral boundary conditions on $C[0, 1]$

$$u''(t) - 4t\, u(1/2) - (2t + 1)\, u(1) = 1 - 5t, \tag{36}$$

$$u(0) = \frac{3}{2} \int_0^1 x^2 u''(x)\, dx, \quad u'(0) = -\frac{1}{7} \int_0^1 (3x + 2) u''(x)\, dx, \tag{37}$$

is correct and the unique solution of (36), (37) is given by

$$u(t) = t^2 - t + 1. \tag{38}$$

Proof. First we rewrite the boundary conditions (37) in the form

$$\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} = \begin{pmatrix} 3/2 & 0 \\ 0 & -1/7 \end{pmatrix} \begin{pmatrix} \int_0^1 x^2 u''(x)\, dx \\ \int_0^1 (3x+2) u''(x)\, dx \end{pmatrix}. \qquad (39)$$

If we compare (36), (39) with (14), it is natural to take $X = Y = C[0, 1]$, $Au(t) = \hat{A}u\ (t) = u''(t)$, $D(A) = C^2[0, 1]$, The operator A, by Lemma 3.1, is closed. Furthermore A is a maximal operator, $X_A = (C^2[0, 1], ||\cdot||_{XA})$, where $||u(t)||_{XA} = ||u(t)|| + ||u''(t)||$. The set $z = (1, t)$ constitute a basis of ker A. As the operator $\hat{A}$ it is natural to take $\hat{A}u\ (t) = Au(t) = u''(t)$, $D(\hat{A}) = \{u(t) \in$ $\in D(A) : u(0) = u'(0) = 0\}$. The initial problem $\hat{A}u\ (t) = f(t)$ is correct and has the unique solution $\hat{A}^{-1}f(t) = \int_0^t (t - x)f(x)dx$. By comparing (36), (39) with (14), it is natural to take $g_1 = 4t$, $g_2 = 2t + 1$, $g = (g_1, g_2) = (4t, 2t + 1)$,

$$f = 1 - 5t,\ N = \begin{pmatrix} 3/2 & 0 \\ 0 & -1/7 \end{pmatrix},$$

$$\Phi(u) = \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \end{pmatrix},\ \Psi(u) = \begin{pmatrix} \Psi_1(u) \\ \Psi_2(u) \end{pmatrix},\ F(Au) = \begin{pmatrix} F_1(Au) \\ F_2(Au) \end{pmatrix},\ \text{where}$$

$\Phi_1(u) = u(0)$, $\Phi_2(u) = u'(0)$, $\Psi_1(u) = u(1/2)$, $\Psi_2(u) = u(1)$,

$$F_1(Au) = \int_0^1 x^2 u''(x)\, dx,\ F_2(Au) = \int_0^1 (3x+2) u''(x)\, dx.\ \ \text{Then}\ \ F_1(f) =$$

$$= \int_0^1 x^2 f(x)\, dx,\ F_2(f) = \int_0^1 (3x+2) f(x)\, dx.\ \text{It is evident that the vector } z = (1, t)$$

is biorthogonal to (Φ_1, Φ_2). and that $/F_1(f)/ \leq ||f||$, $/F_2(f)/ \leq 5 //f//$ for all $f \in C[0, 1]$. So $F_1, F_2 \in C^*[0, 1] = X^*$. Because of $/\Psi(u)/ = /u(t_0)/ \leq$ $\leq ||u(t)|| \leq ||u(t)// + ||u''(t)// = ||u(t)//_{XA}$, we conclude that $\Psi_1, \Psi_2 \in [X_A]^*$. In the same way is proved that $\Phi_1 \in [X_A]^*$. Now by using Inequality (1), for all $u \in C^2[0, 1]$ we obtain $|\Phi_2(u)| = |u'(0)| \leq ||u'(t)|| \leq ||u''(t)|| + 2||u(t)|| \leq$ $\leq 2(||u''(t)|| + ||u(t)||) = 2||u(t)||_{XA}$.

This proves that $\Phi_2 \in [X_A]^*$. The conditions 1.1–1.6 of Remark 2.8 hold true and so we can apply Theorem 2.6. Now we calculate $\hat{A}^{-1}g_1(t) =$

$$= \int_0^t (t-x) 4x\, dx = 2t^3/3, \hat{A}^{-1}g_2(t) = \int_0^t (t-x)(2x+1)\, dx = 1/6 t^2 (2t+3),$$

$$\Psi\left(\hat{A}^{-1}g\right) = \begin{pmatrix} \Psi_1\left(\hat{A}^{-1}g_1\right) & \Psi_1\left(\hat{A}^{-1}g_2\right) \\ \Psi_2\left(\hat{A}^{-1}g_1\right) & \Psi_2\left(\hat{A}^{-1}g_{12}\right) \end{pmatrix} = \begin{pmatrix} \left(\hat{A}^{-1}g_1\right)(1/2) & \left(\hat{A}^{-1}g_2\right)(1/2) \\ \left(\hat{A}^{-1}g_1\right)(1) & \left(\hat{A}^{-1}g_2\right)(1) \end{pmatrix} =$$

$$= \begin{pmatrix} 1/12 & 1/6 \\ 2/3 & 5/6 \end{pmatrix}, \quad \Psi(z) = \begin{pmatrix} \Psi_1(z_1) & \Psi_1(z_2) \\ \Psi_2(z_1) & \Psi_2(z_2) \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1 & 1 \end{pmatrix},$$

$$F(g) = \begin{pmatrix} F_1(g_1) & F_1(g_2) \\ F_2(g_1) & F_2(g_2) \end{pmatrix} = \begin{pmatrix} 1 & 5/6 \\ 8 & 15/2 \end{pmatrix}, \quad L_2 = I_m - \Psi\left(\hat{A}^{-1}g\right) - \Psi(z)NF(g)$$

$$= \begin{pmatrix} -1/84 & -37/42 \\ -43/42 & -1/84 \end{pmatrix}.$$ Since det $L_2 \neq 0$, by Theorem 2.6, the problem (36),

(37) is correct. The inverse matrix $L_2^{-1} = \dfrac{1}{303}\begin{pmatrix} 4 & -296 \\ -344 & 4 \end{pmatrix}$. For $f(t) = 1-5t$

we compute

$$\hat{A}^{-1} f(t) = \int_0^t (t-x)(1-5x)\,dx = 1/6t^2(3-5t),$$

$$F_1(f) = \int_0^t x^2(1-5x)\,dx = -11/12, \quad F_2(f) = \int_0^1 (3x+2)(1-5x)\,dx = -13/2,$$

$F(f) = col(-11/12, -13/2)$, $\Psi(\hat{A}^{-1}f) = col(\Psi_1(\hat{A}^{-1}f), \Psi_2(\hat{A}^{-1}f)) = (1/48, -1/3)$.

From (17) we get $p = [\hat{A}^{-1}g + zNF(g)]\ L_2^{-1} = -1/303(4(28t^3 + 43t^2 - 91t + 106), 196t^3 - 2t^2 - 334t + 439)$.

From (16), by Theorem 2.6, we obtain the unique solution

$$u(t) = \frac{1}{6}t^2(3-5t) - \frac{1}{303}\left(4\left(28t^3 + 43t^2 - 91t + 106\right), 196t^3 - 2t^2 - 334t + 439\right) \times$$

$$\times \begin{pmatrix} 1/48 \\ -1/3 \end{pmatrix} + \left[(1,t) - \frac{1}{303}\left(4\left(28t^3 + 43t^2 - 91t + 106\right), 196t^3 - 2t^2 - 334t + 439\right) \times \right.$$

$$\times \left. \begin{pmatrix} 1 & 1/2 \\ 1 & 1 \end{pmatrix} \right] \cdot \begin{pmatrix} 3/2 & 0 \\ 0 & -1/7 \end{pmatrix} \begin{pmatrix} -11/12 \\ -13/2 \end{pmatrix}$$ which yields (38).

Example 3.2. The next problem with differential equation and non-local integral boundary conditions on $C[0, 1]$

$$u''(t) = -\pi^2 \cos(\pi t), \tag{40}$$

$$u(0) = \frac{3}{2}\int_0^1 x^2 u''(x)\,dx, \ u'(0) = \frac{3}{4}\int_0^1 (2x+3)u''(x)\,dx,$$

is correct and the unique solution of (40) is given by

$$u(t) = \cos(\pi t) + 3t + 2. \tag{41}$$

Proof. First we rewrite the boundary conditions (40) in the form

$$\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/4 \end{pmatrix} \begin{pmatrix} \int_0^1 x^2 u''(x)\,dx \\ \int_0^1 (2x+3)u''(x)\,dx \end{pmatrix}. \tag{42}$$

If we compare (40), (42) with (32), it is natural to take the spaces X, Y, X_A, the operators A, $\hat{A}$, $\hat{A}^{-1}$, the vectors z, Φ_1, Φ_2, F_1 as in Example 3.1,

$$f = -\pi^2 \cos(\pi t), N = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/4 \end{pmatrix}, F_2(Au) = \int_0^1 (2x+3)u''(x)\,dx. \qquad \text{Then}$$

$F_2(f) = \int_0^1 (2x+3)f(x)\,dx.$ Note that for all $f(t) \in C[0,1]$ it follows that $|F_2(f)| \leq 5\|f(t)\|$ and so $F_2 \in C^*[0,1]$. For $f(t) = -\pi^2\cos(\pi t)$ we compute

$$\hat{A}^{-1}f(t) = -\pi^2 \int_0^t (t-x)\cos(\pi x)\,dx = \cos(\pi t) - 1, F_1(f) = -\pi^2 \int_0^1 x^2 \cos(\pi x)\,dx = 2,$$

$F_2 = -\pi^2 \int_0^1 (2x+3)\cos(\pi x)\,dx = 4.$ By Corollary 2.7, the problem (40) is correct. Next as in Example 3.1 is proved that $F_2 \in X^*$. From (33), by Corollary 2.7, we obtain the unique solution

$$u(t) = \cos(\pi t) - 1 + (1,t)\begin{pmatrix} 3/2 & 0 \\ 0 & 3/4 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix},$$

which yields (41).

Example 3.3. The next problem with loaded integro-differential equation and nonlocal integral boundary conditions on $C[0,1]$

$$u''(t) - 4\pi^2 \cos(\pi t)u(1) + \frac{\pi}{6}\sin(\pi t)\int_0^1 u(x)\,dx = 6\pi^2 \cos(\pi t) + (3\pi^2 - 1)\sin(\pi t),$$

$$u(0) = -\frac{2}{\pi^2}\int_0^1 \cos(\pi x)u''(x)\,dx, \ u'(0) = -\frac{2}{\pi}\int_0^1 \sin(\pi x)u''(x)\,dx, \tag{43}$$

is correct and the unique solution of (43) is given by

$$u(t) = 2\cos(\pi t) - 3\sin(\pi t).\tag{44}$$

Proof. First we rewrite the boundary conditions (43) in the form

$$\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} = \begin{pmatrix} -2/\pi^2 & 0 \\ 0 & -2/\pi \end{pmatrix} \begin{pmatrix} \int_0^1 \cos(\pi x) u''(x)\, dx \\ \int_0^1 \sin(\pi x) u''(x)\, dx \end{pmatrix}.\tag{45}$$

If we compare (43), (4.5) with (14), it is natural to take the spaces X, Y, X_A, the operators A, $\hat{A}$, $\hat{A}^{-1}$, the vectors z, Φ_1, Φ_2 as in Ex. 3.1, $g_1 = 4\pi^2\cos(\pi t)$, $g_2 = -(\pi/6)\sin(\pi t)$, $f = 6\pi^2\cos(\pi t) + (3\pi^2 - 1)\sin(\pi t)$,

$$F(Au) = \begin{pmatrix} F_1(Au) \\ F_2(Au) \end{pmatrix}, \quad N = \begin{pmatrix} -2/\pi^2 & 0 \\ 0 & -2/\pi \end{pmatrix},$$

$$F_1(Au) = \int_0^1 \cos(\pi x) u''(x)\, dx, \quad F_2(Au) = \int_0^1 \sin(\pi x) u''(x)\, dx.\ \text{Then}$$

$$F_1(f) = \int_0^1 \cos(\pi x) f(x)\, dx, \quad F_2(f) = \int_0^1 \sin(\pi x) f(x)\, dx.\ \text{Further, we take}$$

$$\Psi_1(u) = u(1), \quad \Psi_2(u) = \int_0^1 u(x)\, dx.\ \text{It is evident that } |F_i(f)| \leq ||f(t)||\text{ for all}$$

$f(t) \in C[0,1]$, $i = 1, 2$. Such $F_i \in C^*[0, 1]$, $i = 1,2$. We proved in Ex. 3.1 that the functional defined by $\Psi(u) = u(1)$ belongs to X_A^*. So $\Psi_1 \in X_A^*$. From Theorem 2.3 it follows that $\Psi_2 \in C^*[0, 1]$. Then for all $u(t) \in X_A$ we get $|\Psi_2(u)| \leq k||u(t)|| \leq k(||u(t)|| + ||u''(t)||) = k||u(t)||_{XA}$. So $\Psi_2 \in X_A^*$.

Further we calculate $\hat{A}^{-1}g_1(t) = 4\pi^2 \int_0^t (t-x)\cos(\pi x)\, dx = 4 - 4\cos(\pi t)$,

$$\hat{A}^{-1}g_2(t) = -\frac{\pi}{6}\int_0^t (t-x)\sin(\pi t)\, dx = \frac{\sin(\pi t)}{6\pi} - \frac{t}{6}, \quad \Psi_1(\hat{A}^{-1}g_1) = (\hat{A}^{-1}g_1)(1) = 8,$$

$$\Psi_1(\hat{A}^{-1}g_2) = (\hat{A}^{-1}g_2)(1) = -1/6, \quad \Psi_2(\hat{A}^{-1}g_1) = \int_0^1 \hat{A}^{-1}g_1(x)\, dx = 4,$$

$$\Psi_2(\hat{A}^{-1}g_2) = \int_0^1 \hat{A}^{-1}g_2(x)\, dx = -1/12.\ \text{So}$$

$$\Psi\left(\hat{A}^{-1}g\right)=\begin{pmatrix}\Psi_1\left(\hat{A}^{-1}g_1\right) & \Psi_1\left(\hat{A}^{-1}g_2\right)\\ \Psi_2\left(\hat{A}^{-1}g_1\right) & \Psi_2\left(\hat{A}^{-1}g_2\right)\end{pmatrix}=\begin{pmatrix}8 & -1/6\\ 4 & -1/12\end{pmatrix}.$$ Also we compute

$$F\left(g\right)=\begin{pmatrix}F_1\left(g_1\right) & F_1\left(g_2\right)\\ F_2\left(g_1\right) & F_2\left(g_2\right)\end{pmatrix}=\begin{pmatrix}2\pi^2 & 0\\ 0 & -\pi/12\end{pmatrix},\ \Psi\left(z\right)=\begin{pmatrix}\Psi_1\left(z_1\right) & \Psi_1\left(z_2\right)\\ \Psi_2\left(z_1\right) & \Psi_2\left(z_2\right)\end{pmatrix}=$$

$$=\begin{pmatrix}1 & 1\\ 1 & 1/2\end{pmatrix},\ L_2=I_m-\Psi\left(\hat{A}^{-1}g\right)-\Psi\left(z\right)NF\left(g\right)=\begin{pmatrix}-3 & 0\\ 0 & \left(3\pi^2-1\right)/\left(3\pi^2\right)\end{pmatrix}.$$

Since $\det L_2 \neq 0$, by Theorem 2.6, the problem (43) is correct. For $f(t) =$
$= 6\pi^2\cos(\pi t) + (3\pi^2-1)\sin(\pi t)$ we compute $F_1\left(f\right)=3\pi^2, F_2\left(f\right)=\dfrac{3\pi^2-1}{2},$

$$\hat{A}^{-1}f\left(t\right)=-6\cos\left(\pi t\right)+\frac{1-3\pi^2}{\pi^2}\sin\left(\pi t\right)+\frac{3\pi^2-1}{\pi}t+6.$$

$$\Psi\left(\hat{A}^{-1}f\right)=col(\Psi_1\left(\hat{A}^{-1}f\right),\Psi_2\left(\hat{A}^{-1}f\right)=col\left(\frac{3\pi^2+12\pi-1}{\pi},\frac{3\pi^4+12\pi^3-13\pi^2+4}{2\pi^3}\right).$$

Next we find $L_2^{-1}=\begin{pmatrix}-1/3 & 0\\ 0 & 3\pi^2/\left(3\pi^2-1\right)\end{pmatrix},\ \ p=\left(\dfrac{4\cos\left(\pi t\right)}{3},\dfrac{\pi\sin\left(\pi t\right)}{2\left(3\pi^2-1\right)}\right).$

Sustituting these values into (16), we obtain the unique solution (44).

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