



ВЕСТНИК ПНИПУ. МЕХАНИКА

№ 2, 2021

PNRPU MECHANICS BULLETIN

<https://ered.pstu.ru/index.php/mechanics/index>



DOI: 10.15593/perm.mech/2021.2.01

УДК 517.958:539.3(2)

## CONTRADICTIONS IN THE PLANE CONTACT PROBLEM OF THE THEORY OF ELASTICITY ON THE COMPRESSION OF CYLINDERS IN CONTACT WITH PARALLEL GENERATORS

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### ARTICLE INFO

Received: 4 March 2021

Accepted: 7 June 2021

Published: 12 July 2021

#### Keywords:

displacement, convergence, cylinder, stress, force, load, compression, contact pressure, half-plane, uniformity, isotropy, elasticity.

### ABSTRACT

The inapplicability of the reference formula for determining the convergence of two statically compressed parallel cylinders made of a homogeneous, isotropic and physically linear material has been proved due to a well-known logarithmic feature in the plane classical problem of mechanics of elastic solids. In the special case of the elastic interaction of a cylinder with a half-plane, when one of the radii has an infinite length, it has been found that the convergence also becomes equal to infinity. This paradoxical result contradicts not only the physical and mechanical meaning of the process under study, but also confirms the inadequacy of Flamant model of a simple radial stress state in determining displacements. The authors have proposed an algorithm for eliminating the contradictions based on the solution of Fredholm integral equation of the first kind. In the future, it can be considered as a new fundamental and applied problem of the theory of elasticity, which is of a great importance for a refined assessment of the contact strength and stiffness of the cylindrical parts of load-bearing structures taking into account the general and local deformations (cylindrical rollers, gears, road surfaces, when they are compacted with steel rollers, etc.) on the basis of a flat Flamant calculation scheme considering three stress components and the width of the cylinder contact area previously developed and mathematically approximated by the authors.

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The load-bearing structural elements in the form of cylindrical rollers interacting in contact on a surface of finite dimensions are widely used in various branches of modern mechanical engineering and construction. The typical examples of such parts are: plain bearings; supporting parts of bridges, overpasses and sluice gates; wheels of railway rolling stock, etc. [1–5].

The well-known constructively nonlinear [6] theory of small elastic contact deformations of two physically linear, isotropic and homogeneous circular cylinders is based on the following assumptions (Fig. 1) [1–3, 7–24]:

1) radii  $R_1, R_2$  of the cylindrical bodies are large compared to the  $2a$  size of the contact area, i.e.

$$R_1 \gg 2a, R_2 \gg 2a \quad (1)$$

here  $a$  is half the pressure band width;

2) the cylinders are substantially parallel to the longitudinal axis  $O_1, O_2$  and width  $l \gg 2a$ ;

3) within the limits of assumption (1), the contact pad can be considered as part of the plane tangent to the guide (circle) of the undeformed cylinders at the place of their initial contact in a straight line;

4) there is no friction between the touching surfaces, which are assumed to be absolutely smooth;

5) the contacting elements are pressed against each other by two equal in magnitude and oppositely directed external forces – resultants  $Q$ , distributed over a given length  $l$  of the cylinders as a constant static load

$$P = \frac{Q}{l} = \text{const}, \quad (2)$$

if the equilibrium condition is met [8, 10–15, 17, 18]

$$l \cdot \int_{-a}^a q(y) dy = P \cdot l = Q, \quad (3)$$

here  $q = q(y)$  – reactive boundary stress approximated by the elliptic Hertz function [8, 19–24, 27–31]

$$q(y) = q_0 \sqrt{1 - \frac{y^2}{a^2}}, \quad -a \ll y \ll a, \quad (4)$$

having an extremum [1, 8–13, 17–19, 20, 22]

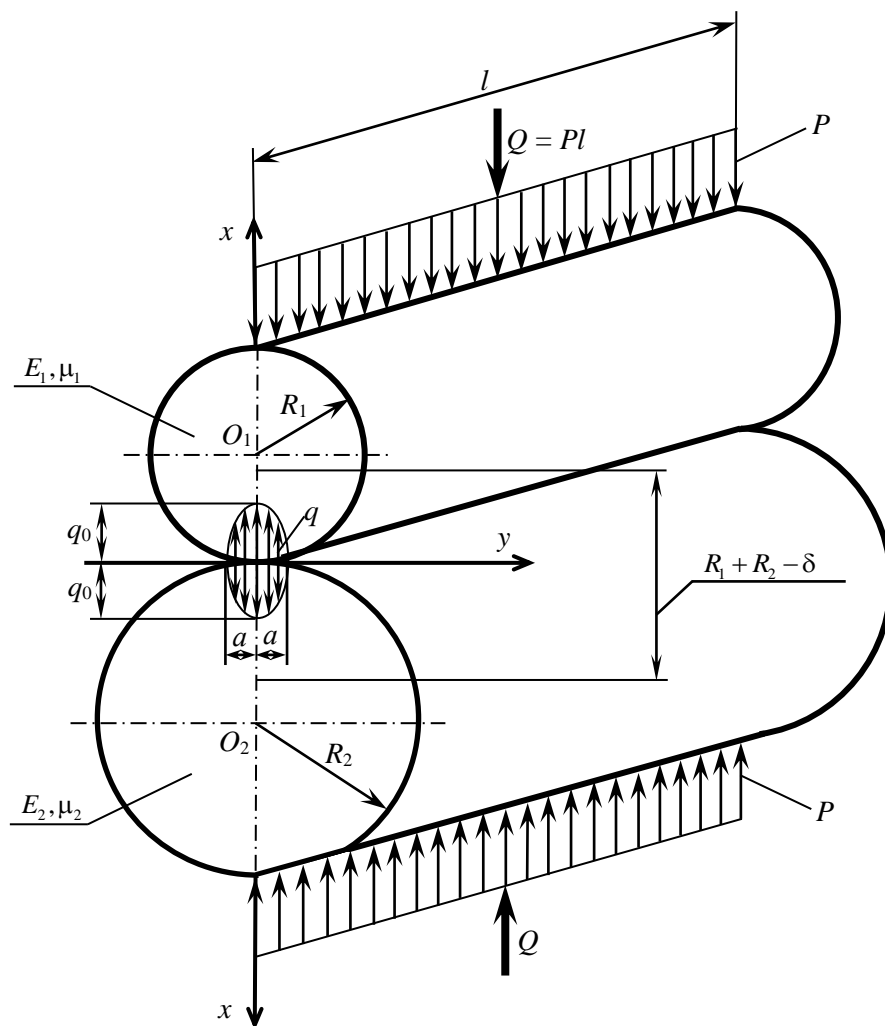


Fig. 1. Design and construction layout of contacting elastic cylinders

$$q_o = \frac{2}{\pi} \cdot \frac{Q}{a \cdot l} = \max. \quad (5)$$

If the cylinders are made of materials with elastic modulus  $E_1, E_2$ , Poisson's ratio are  $\mu_1, \mu_2$ , the formulas  $a, q_o$  and total kinematic displacement  $\delta$  (convergence of axes  $O_1, O_2$ ), have the form [1, 2, 32]:

$$a = 2 \sqrt{\frac{Q}{\pi \cdot l} \cdot \frac{R_1 \cdot R_2}{(R_1 + R_2)} \cdot \left( \frac{1 - \mu_1^2}{E_1} + \frac{1 - \mu_2^2}{E_2} \right)}, \quad (6)$$

$$q_o = \sqrt{\frac{Q}{\pi \cdot l} \cdot \frac{(R_1 + R_2)}{R_1 \cdot R_2} \cdot \left( \frac{1 - \mu_1^2}{E_1} + \frac{1 - \mu_2^2}{E_2} \right)^{-1}}, \quad (7)$$

$$\delta = \frac{2 \cdot Q}{\pi \cdot l} \cdot \left[ \frac{(1 - \mu_1^2)}{E_1} \cdot \left( \ln \frac{2 \cdot R_1}{a} + 0,407 \right) + \frac{1 - \mu_2^2}{E_2} \cdot \left( \ln \frac{2 \cdot R_2}{a} + 0,407 \right) \right], \quad (8)$$

The relations (6)–(8) are used in the quantitative assessment of the bearing capacity of the cylindrical system of Figure 1. From a practical point of view, these are the design calculations for the contact strength and stiffness of friction and gear gears, roller parts of bridge supports and other critical elements of engineering structures.

At the same time, it should be noted that the determination of the  $\delta$  displacement according to the reference-normative formula (8) has a significant mechanical and mathematical incorrectness [2, 7], like in the classical Fla-

mant problem [2, 7, 9, 10, 15–24, 29, 34] on the action of the concentrated-distributed force  $P = \text{const}$  (See Fig. 1) on the elastic isotropic half-plane underlying the relation (8), where the displacement is calculated relative to a point sufficiently distant from the point of contact, the position of which is unknown.

The centers of curvature  $O_1$  and  $O_2$  (See Fig. 1) are taken as such points in the analytical dependence (8) under the hypothetical assumption that the  $\delta$  parameter is determined only by the general deformations of the cylinders, without taking into account the contact components, which, according to [35], can represent a significant part (from 30 to 90 %) in the total balance of elastic displacements of the contacting parts.

The specified uncertainty (multivariance) in the choice of the fixed point coordinate when determining the displacements directed perpendicular to the boundary of the half-plane is a consequence of the general logarithmic feature of Flamant physical and mathematical model [7, 9, 10, 19, 27, 29]. In the same connection, the author of the classical fundamental publication [7] states that it is possible to determine only the stresses in the parallel contacting cylinders on the basis of Flamant solution, and the calculation of displacements in this case is not possible. Thus, it can be stated that the evaluation of the contact stiffness according to the formula (8) will not adequately characterize the deformed state of the cylinders. Another negative consequence of the presence of the  $\ln$  logarithm in the dependence (8) is shown for its special case when one of the radii, for example, is  $R_2 = \infty$ . At the same time, we will have a fairly common engineering problem in the design calculations about the contact of a compressible cylinder with an elastically deformable half-plane (Fig. 2) [1–5].

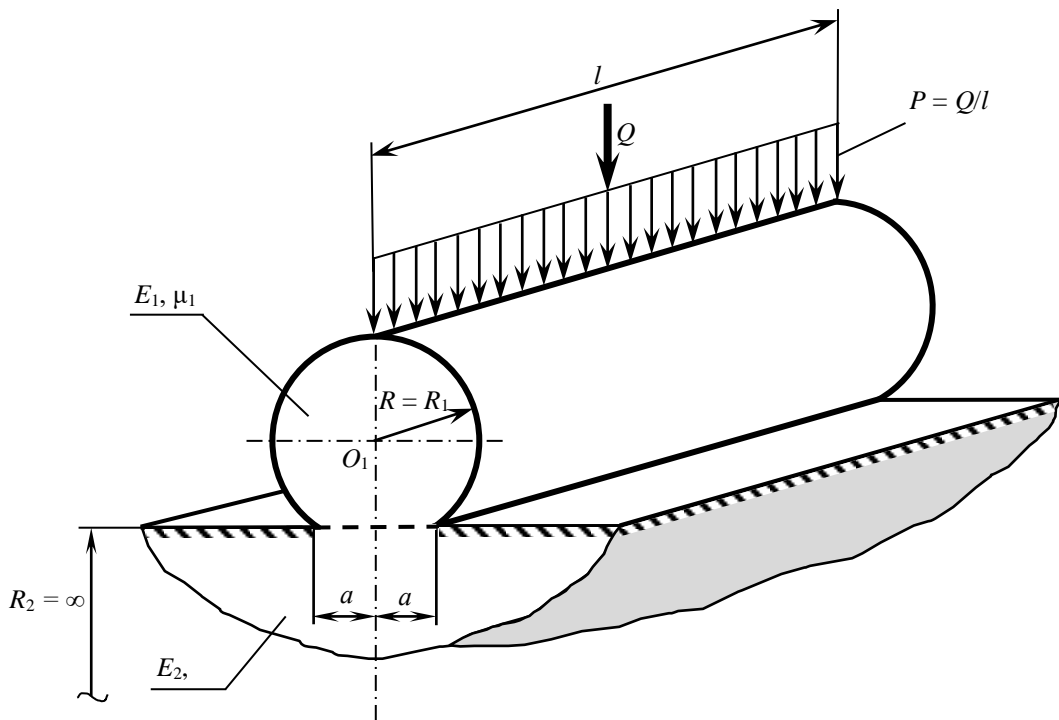


Fig. 2. A model of a cylinder pressure on a plane

Given condition  $R_2 = \infty$  and equality  $R_1 = R$ , we get the final value in accordance with (6) for linear size  $a$

$$a = 2 \sqrt{\frac{Q \cdot R}{\pi \cdot l} \cdot \left( \frac{1 - \mu_1^2}{E_1} + \frac{1 - \mu_2^2}{E_2} \right)}. \quad (9)$$

Substitution (9) in (8) at  $R_2 = \infty$  results in the paradoxical solution  $\delta = \infty$  which contradicts the physical meaning of the contact problem under consideration and confirms the unacceptability of Flamant mathematical model in determining the  $\delta$  displacements [7, 28].

The erroneous result  $\delta = \infty$  does not correspond to the basic fundamental Boussinesq problem of [12, 15–18, 25, 27–29, 30, 31, 34, 36] on a perpendicularly directed concentrated force on an elastic half-space in which the above contradictions are not present.

A fundamentally new displacement formula has been obtained in the paper [37] guided by the classical interpretation of the plane linear elastic deformation [2, 8, 9, 19–22, 26, 28–30] and the refined innovative solution of Flamant

problem [37], which includes three stresses (compared to one radial component [19, 20]) and the  $a$  parameter

$$v_h = v_h(y) = \frac{2 \cdot P \cdot (1 - \mu^2)}{3 \cdot \pi \cdot E} \cdot \left( \frac{a}{y} \right)^2 \quad (10)$$

half-plane boundaries in an unlimited range of the variable variation  $y$ .

In contrast to the incorrect logarithmic dependence [9, 19–21, 28, 29, 31]

$$v_{hf} = v_{hf}(y) = \frac{2 \cdot P \cdot (1 - \mu^2)}{\pi \cdot E} \cdot \ln \frac{l_k}{|y|}, \quad (11)$$

containing the distance to an arbitrary point  $K$  (Fig. 3) and approximating only the relative value of the displacements  $v_{hf}$  on a closed interval  $-l_k \leq y \leq l_k$ , the formula (10) derived in [37] allows determining the absolute draft of the  $x = 0$  boundary of the half-plane without reference to the parameter  $l_k$  on a theoretically infinite interval  $-\infty \leq y \leq \infty$ .

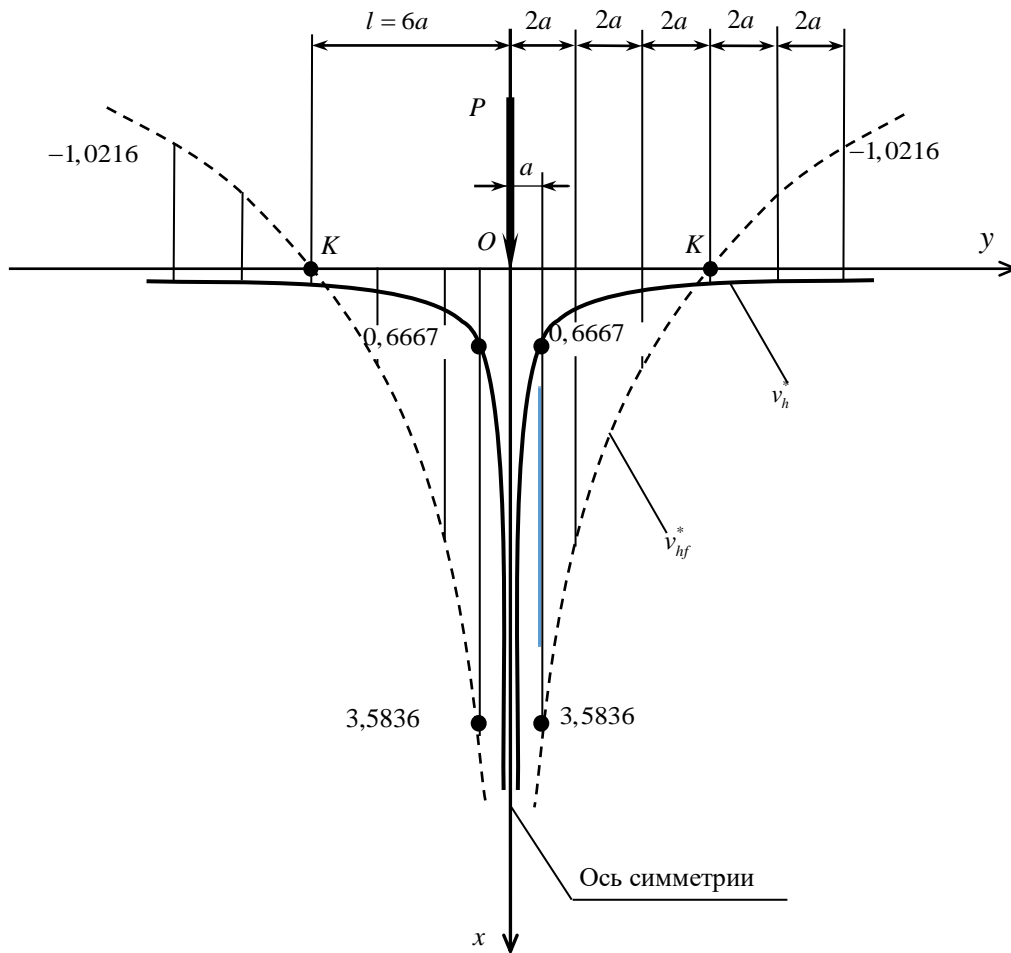


Fig. 3. General view of the changes in the  $v_h^*$ ,  $v_{hf}^*$  functional dependencies:  $v_h^*(y)$  is a solid curve according to the new formula (12) [37];  $v_{hf}^*(y)$  the dashed line is in accordance with the Flamant solution (13) [8, 19–21, 28, 29, 31]

The behavior of functions (10) and (11) is illustrated in the dimensionless forms

$$v_h^*(y) = v_h(y) \cdot \frac{\pi \cdot E}{P \cdot (1 - \mu^2)} = \frac{2}{3} \cdot \left(\frac{a}{y}\right)^2, \quad (12)$$

$$v_{hf}^*(y) = v_{hf}(y) \cdot \frac{\pi \cdot E}{P \cdot (1 - \mu^2)} = 2 \cdot \ln \frac{l_k}{|y|} \quad (13)$$

in Fig. 3 using the numerical information of the table.

The values of the functional relations (12), (13) when  $l_k = 6a$

y	0	±a	±2a	±4a	±6a	±8a	±10a	±∞
$v_h^*$	∞	0,6667	0,1667	0,0416	0,0186	0,0104	0,0066	0
$v_{hf}^*$	∞	3,5836	2,1972	0,8110	0	-0,5751	-1,0216	-∞

A mechanical system in which a local uniformly distributed stationary  $P$  force acts on an elastic isotropic medium (see Fig. 1–3) should be considered as abstract, not reflecting the actual possible conditions. However, using the formally idealized mathematical solution (10), we proceed to the real simulation of the reactive load  $q = q(y)$  that occurs between contacting elastic-deformable bodies (see Fig. 1).

That is why we present the following the Fredholm equation of the first kind [9, 20, 23, 27–29, 38, 39] with an unknown  $q(y)$  function under the sign of a certain integral by analogy with the developed theory of calculating the draft of the belt foundation [37] and guided by [8, 9, 22, 24, 28-31, 38-40] to answer the question:

$$v_r = v_{r1} + v_{r2} \equiv -\frac{2 \cdot a^2}{3 \cdot \pi} \cdot \left( \frac{1 - \mu_1^2}{E_1} + \frac{1 - \mu_2^2}{E_2} \right) \times \int_{-a}^a \frac{q(y) \cdot dy}{(t - y)^2} = \delta - \frac{1}{2} \cdot \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \cdot t^2. \quad (14)$$

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here  $q = q(y), q_0, a, \delta$  are the desired physical and geometric features of the interaction process of two round cylindrical elements in compliance with the obvious requirements

$$-a \leq y \leq a, q(\pm a) = 0, q(0) = q_0 = \max \quad (15)$$

and fulfilling the condition (3);  $t$  is an auxiliary variable that varies within the range of  $-a \leq t \leq a$  [2, 35].

As the paper [37] authors note, comparing with an absolutely-idealized original (10) in which the directions of  $P$  force and  $v_h > 0$  movement coincide (see Fig. 3) in formula (14), the "minus" indicates the opposite effect of the contact pressure  $q(y)$  and  $v_{h1}, v_{h2}$  kinematic components within each cylinder (see Fig. 1, 3).

The analysis of the conducted computational and theoretical studies allows drawing the following conclusions.

**Conclusions**

1. The impossibility [7, 8, 28] of applying the reference dependency is confirmed and proved (8) [1, 2, 32] for calculating the convergence of the cylinders (see Fig. 1, 2) due to the logarithmic singularity noted in the well-known fundamental mechanical and mathematical works [7, 9, 15, 19, 28].
2. For the same reason (see point 1), it is proved that Flamant basic model [8, 9, 19-23, 28-32, 36] is incorrect to solve any plane problems of the theory of elasticity (see Fig. 3) related to the assessment of the contact stiffness [35] of the parallel cylindrical parts of load-bearing structures.
3. In order to completely eliminate the contradictions considered in the article, a linear integral equation (14) is formulated that allows determining: the reactive pressure function  $q(y)$  and its extremum  $q_0 = \max$ ; the half-width  $a$  of the contact area and the total mutual displacement  $\delta$  of the points  $O_1, O_2$  (see Fig. 1).

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